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Convergence Analysis of Newton-Raphson Method in Feasible Power-Flow for DC Network

Zhangjie Liu, Xin Zhang, Mei Su, Yao Sun, Hua Han, and Peng Wang

Abstract—**The Newton-Raphson (NR) method is a powerful tool in nonlinear equations. However, a well-known disadvantage of this method is that the initial iteration value must be chosen sufficiently close to a true solution in order to guarantee its convergence. The Kantorovich theorem is virtually the only known sufficiency condition for convergence of NR method, and gives very conservative bounds. This letter analyzes the convergence of the NR method in feasible power-flow for direct-current (DC) network, and an explicit convergence condition is obtained. Comparing with the existing results, the proposed convergence condition is more concise and less conservative. Moreover, according to the proposed condition, one can easily set the initial iteration value to guarantee the convergence of the NR method. Case studies verify the correctness of the presented analysis.**

Index Terms—**DC network, power-flow equation, Newton-Raphson method, convergence analysis.**

I. Introduction

With the increasing of the direct-current (DC) generations and loads, the application requirements of DC distributions are increasing [1], [2]. In the DC type of network, the load is usually connected to the DC bus through a power Electronics converter, which makes the load behave as constant power load (CPL) [3], [4]. The power-flow equation of DC network becomes nonlinear due to the presence of the CPLs, which complicates the problem [4].

For the nonlinear power-flow equation, the key can be summarized as the following two questions:

1) Under what conditions should the power-flow equation admit a unique feasible solution (i.e., real positive solution)?
2) How to choose the initial iterative value can make the Newton-Raphson (NR) method converge to the feasible solution?

The first problem is the existence and uniqueness of the power-flow equation, which is the prerequisite for a stable DC network. For the first problem, several results based on fixed-point theorem [5]–[8] and nested interval theorem [9] are obtained. The second problem is the convergence of the NR method. It is well-known that the NR method has a quadratic convergence rate, but it converges only when the iterative initial value is close enough to a solution. Most existing studies are based on numerical simulation [10]. One of the few analytical results based on the Kantorovich's theorem is presented in [11], which is general and can be applied to both grid-connected and island operation mode. However, in the grid-connected mode, the results in [11] are complex and conservative. This letter focuses on the second problem for the DC microgrid where the generators are constant voltage and loads are CPLs, and aims to give a more concise and less conservative convergence conditions.

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II. Notations and Preliminary Results

The next parts employ the following two definitions and two lemmas:

Definition 1: For a real vector $h = [h_1 h_2 \ldots h_m]^\top (h_i \neq 0)$, we denote $[[h]]= diag\{h_i\}$ and $h^n = [\, h_1^n \; h_2^n \; \cdots \; h_m^n \,]^\top$, where n is an integer. Define $1_m, 0_m$ as the m-dimensional vector which all entries are 1 and 0, respectively. Denote O as the m-by-m zero matrix.

Definition 2: Denote $A > B$, if the entries of A$-$B are all positive. Matrix A (or a vector) is called positive if its entries are all positive.

Definition 3: Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the eigenvalues of a square matrix $A \in \mathcal{R}^{m\times m}$. Then its spectral radius $\rho(A)$ is

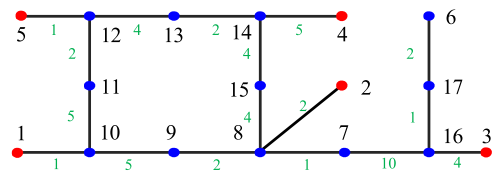

Fig. 1. The structure of the simulation DC microgrid.

defined as $\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \cdots |\lambda_m|\}$. Denote $\|A\| = \max_{1\le i\le m}(\sum_{j=1}^{m}|a_{ij}|)$.

Lemma 1: Let $A > B > O$, then,$\rho(A) > \rho(B)$ [9].

Lemma 2: Let $A > O$, if $\rho(A) < 1$, then, $(I - A)^{-1} > O$ [9].

III. Existence Conditions for Power-Flow Equation of DC Microgrid

Consider a DC network with n generator nodes and m load nodes, and all the loads are assumed as CPLs. The KCL equation of the DC network can be expressed as

$$\begin{bmatrix} i_G \\ i_L \end{bmatrix} = \begin{bmatrix} B_{GG}\ B_{GL} \\ B_{LG}\ B_{LL} \end{bmatrix} \begin{bmatrix} u_G \\ u_L \end{bmatrix} = B \begin{bmatrix} u_G \\ u_L \end{bmatrix}$$
$$\times \begin{cases} i_G = \left[i_1\ i_2 \cdots\ i_n \right]^\top, u_G = \left[u_1\ u_2 \cdots\ u_n \right]^\top \\ i_L = \left[i_{n+1}\ i_{n+2} \cdots\ i_{n+m} \right]^\top, \\ u_L = \left[u_{n+1}\ u_{n+2} \cdots\ u_{n+m} \right]^\top \end{cases} \tag{1}$$

where i_G, u_G, i_L and u_L are the vector of generator currents, voltages, load currents and voltages, respectively. The voltage vector of the generator is a known constant. Matrix B is the equivalent admittance matrix of the transmission network. For the load nodes, its voltage and current yields to

$$[\![u_L]\!]\, i_L = -p, p = \left[p_{n+1}\ p_{n+2} \cdots\ p_{n+m} \right]^\top \tag{2}$$

where p_i is the CPL' power of the i^{th} node. Combining (1) with (2), the following can be obtained.

$$B_{LG}u_G + B_{LL}u_L = -[\![p]\!]\, u_L^{-1} \tag{3}$$

Multiplying byB_{LL}^{-1}, the power-flow equation of DC network can be obtained as the following

$$u_L = -B_{LL}^{-1}B_{LG}u_G - B_{LL}^{-1}[\![p]\!]\, u_L^{-1} \tag{4}$$

In equation (4), u_G is a known positive constant vector. Denote $u_G^* = -B_{LL}^{-1}B_{LG}u_G, x = [\![u_G^*]\!]^{-1}u_L, A = [\![u_G^*]\!]^{-1}B_{LL}^{-1}[\![u_G^*]\!]^{-1}[\![p]\!]$. According to the results in [6], we obtain $u_G^* > 0_m$ and $A > O$. Substituting (5) into (4), the power-flow equation is obtained as

$$f(x) \triangleq x + Ax^{-1} - 1_m = 0_m \tag{5}$$

The results about the solvability of (6) can be found in [9], [6], and the main result can be summarized as the following.

Lemma 3. (See Theorem 2 and 4 in [9]): Equation (6) admits a unique solution in the interval $D = \{x | 0.5 \times 1_m < x < 1_m\}$

TABLE I
Parameters of Cases

	CPL (kW)	x_0	$4\|A\|$	h
Case 1	[11 11 11 11 11 11 22 22 22 22 22 22]	$0.6\times 1_m$	0.96	1.81
Case 2	[6 6 6 6 6 6 12 12 12 12 12 12]	1_m	0.01	0.01
Case 3	[6 6 6 6 6 6 12 12 12 12 12 12]	$0.55\times 1_m$	0.01	0.01
Case 4	[11 11 11 11 11 11 22 22 22 22 22 22]	$0.4\times 1_m$	0.96	1.81

if the following holds

$$4\,\|A\| < 1 \tag{6}$$

Then, then (6) has a solution x^* that satisfies

$$\rho\left(A[\![x^*]\!]^{-2}\right) < 1 \tag{7}$$

IV. Convergence Analysis of Newton-Raphson Method

The NR method for nonlinear equation (6) can be expressed as the following

$$x_{k+1} = x_k - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}\left(x_k + Ax_k^{-1} - 1_m\right) \tag{8}$$

A. Convergence Analysis of the Newton-Raphson Method

Assume (7) holds, and denote x^* referring the solution of (6) in the internal D. Based on Lemma 3, Lemma 4 is derived.

Lemma 4: If x_k satisfies $x_k \neq x^*$ and $\rho(A[\![x_k]\!]^{-2}) < 1$, then the following holds for any positive integer i

$$\begin{cases} x_{k+i} > x_{k+i+1} > x^* & \text{(9a)} \\ \rho\left(A[[x_{k+i}]]^{-2}\right) < \rho\left(A[[x_{k+i+1}]]^{-2}\right) \\ \quad < \rho\left(A[[x^*]]^{-2}\right) < 1, & \text{(9b)} \\ f\left(x_{k+i}\right) > 0_m & \text{(9c)} \\ \lim_{i\to\infty} x_{k+i} = x^* & \text{(9d)} \\ \lim_{i\to\infty} \frac{\|x_{k+i+1} - x^*\|}{\|x_{k+i} - x^*\|^2} = \left\| \left(I - A[[x^*]]^{-2}\right)^{-1} A[[x^*]]^{-3} \right\| & \text{(9e)} \end{cases}$$

Proof: Substituting $1_m = x^* + Ax^{*-1}$ into (9), we obtain

$$\begin{aligned} x_{k+1} &= x_k - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}\left(x_k + Ax_k^{-1} - x^* - Ax^{*-1}\right) \\ &= x_k - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}\left(I - A[[x_k]]^{-1}[[x^*]]^{-1}\right)(x_k - x^*) \\ &= x_k - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}\Big(I - A\left[\left[x_k^{-2}\right]\right] \\ &\quad + A\left[\left[x_k^{-2}\right]\right] - A[[x_k]]^{-1}[[x^*]]^{-1}\Big)(x_k - x^*) \\ &= x^* + \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}A[[x_k]]^{-2}[[x^*]]^{-1}(x_k - x^*)^2 \end{aligned} \tag{10}$$

Assume $\rho(A[[x_k]]^{-2}) < 1$, then, $(I - A[[x_k^{-2}]])^{-1}$ is positive. According to (11), $x_{k+1} > x^*$holds as long as $\rho(A[[x_k^{-2}]]) < 1$. Since $x_{k+1} > x^*$, according to Lemma 1, we obtain $\rho(A[[x_{k+1}^{-2}]]) < \rho(A[[x^*]]^{-2}) < 1$ and $x_{k+2} > x^*$, and so on. Thus, $x_{k+i} > x^*$holds for any positive interger i.

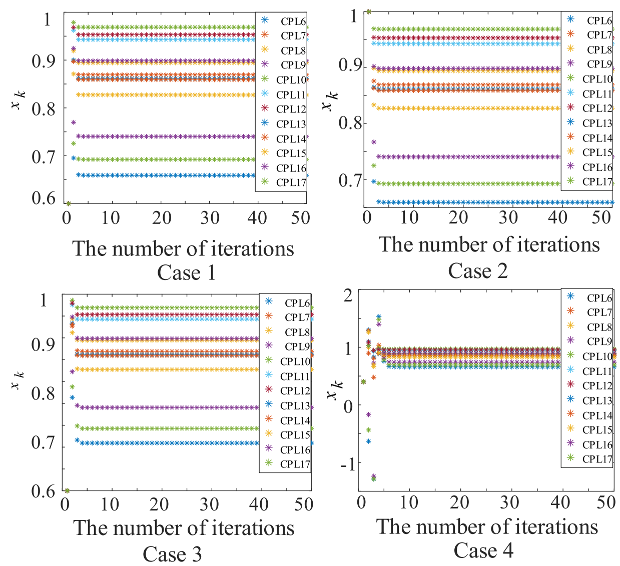

Fig. 2. The iteration process of the NR method. k is the iteration number. x_k is the value of x $(x = u_L \,/\, u_G^*)$ in the k-th iteration.

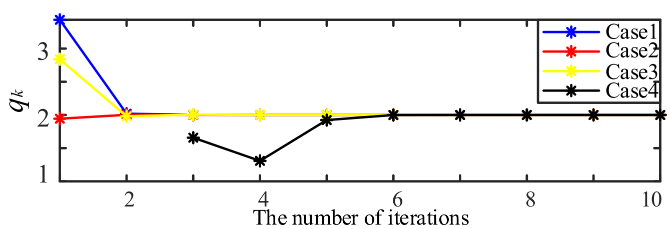

Fig. 3. The curves of q_k of case 1–4.

Considering that

$$
\begin{aligned}
f\left(x_{k+1}\right) &= x_{k+1} + A x_{k+1}^{-1} - 1_m \\
&= x_k - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1} f\left(x_k\right) + A x_{k+1}^{-1} - 1_m \\
&= x_k + A x_k^{-1} - 1_m - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1} f\left(x_k\right) \\
&\quad + A\left(x_{k+1}^{-1} - x_k^{-1}\right) \\
&= \left(I - \left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1}\right) f\left(x_k\right) + A\left(x_{k+1}^{-1} - x_k^{-1}\right) \\
&= -A\left[\left[x_k^{-2}\right]\right]\left(I - A\left[\left[x_k^{-2}\right]\right]\right)^{-1} f\left(x_k\right) + A\left(x_{k+1}^{-1} - x_k^{-1}\right) \\
&= -A\left[\left[x_k^{-2}\right]\right]\left(x_k - x_{k+1}\right) + A\left[\left[x_{k+1}^{-1}\right]\right]\left[\left[x_k^{-1}\right]\right]\left(x_k - x_{k+1}\right) \\
&= A\left[\left[x_k^{-2}\right]\right]\left[\left[x_{k+1}^{-1}\right]\right]\left(x_k - x_{k+1}\right)^2 \qquad (11)
\end{aligned}
$$

Since $x_{k+i} > x^* > 0_m$ holds for any positive interger i, then, (9c) is proved. Since $\rho(A[[x_{k+i}^{-2}]]) < 1$, according to Lemma 2, $(I - A[[x_{k+i}^{-2}]])^{-1} > O$. Thus, (9a) and (9b) are proved. Since (9a) holds, according to the monotone convergence theorem, (9d) is obtained. According to (10), (9e) can be easily obtained. The proof is completed.

From Lemma 4, the main results about convergence of the NR method described in (8) are obtained as the following.

Theorem 1: Denote x_0 as the he initial iteration value. If $4\|A\| < 1$ and $x_0 > 0.5 \times 1_m$, the NR method described in (8) is convergent.

Proof: Assume $4\|A\| < 1$ and $x_0 > 0.5 \times 1_m$, then, we obtain $A[[x_0^{-2}]] < 4A$. According to Lemma 1, we get $\rho(A[[x_0^{-2}]]) < 4\rho(A) < 4\|A\| < 1$. According to Lemma 4, Theorem 1 is proved. The proof is accomplished.

B. Comparing With the Existing Results

For a DC microgrid under the Master-Slave control, the power-flow equation takes the similar form (See equation (1) in [11]). In [11], the main result about the convergence of the NR method can be summarized as the following.

Lemma 5: Define $\rho = \|B_{LL}^{-1}\|, \alpha = \|P\|, \mu = \|B_{LG} u_G + B_{LL} 1_m\|$, and $\alpha\rho < 1$. The NR method starting from $x_0 = 1_m$ is convergent if

$$
h = \frac{\alpha\rho^2\left(1 - \alpha\rho\right)\left(\alpha + \mu\right)}{\left(1 - 2\alpha\rho - \mu\rho\right)^3} < \frac{1}{4} \qquad (12)
$$

the solution lies in the ball $\{x \,|\, \|x - 1_m\| < \delta\}$ with

$$
\frac{\left(\mu + \alpha\right)\rho}{\left(1 - \alpha\rho\right)} < \delta < 1 \qquad (13)
$$

Remark 1: Lemma 5 is one of the main results of [11], which is obtained based on the Kantorovich theorem. The results of [11] are more general and can be applied to both grid-connected and island operation mode. In island operation mode, the proposed conditions in Theorem 1 are concise and less conservative. According to Theorem 1, the NR method described in (9) is convergent if $4\|A\| < 1$ and $x_0 > 0.5 \times 1_m$. Thus, one can easily choose the x_1 to guarantee the convergence of the NR method.

V. Case Study

To verify the presented analyses, we simulate a meshed DC microgrid with 5 DGs and 12 CPLs which is shown in Fig 1. The red and blue points represent DGs and CPLs, respectively. The black line represents the cables. The green numbers are the resistances of cables, and the black numbers are the identifiers of nodes.

To verify the presented analyses, four cases are tested. In all cases, the voltages are $u_1 = u_2 = u_3 = 1$ kV and $u_4 = u_5 = 1.1$ kV. The other parameters are presented in the following.

According to Theorem 1, the NR method is convergent if $4\|A\| < 1$ and$x_0 > 0.5 \times 1_m$. Table I shows that case 1–3 satisfy Theorem1, and only case 2 satisfies the condition proposed in [11]. The simulation results are in Fig. 2.

The iteration processes of the NR method described in (8) are presented in Fig. 2. The curves in different color refer to the voltage ratio $(u_L \,/\, u_G^*)$ of CPL nodes. The results show that the NR methods in case 1–3 are all convergent, which verifies the correctness of the proposed convergence conditions. Case 1 and 3 do not satisfy the condition proposed in [11], which implies that the proposed convergence conditions are less conservative. In case 4, the initial value does not satisfy (15), so $\{x_k\}$ is divergent in the first four steps.

To determine the order of convergence of $\{x_k\}$, we define the sequence $q_k = \log(\frac{\|x_{k+3}-x_{k+2}\|}{\|x_{k+2}-x_{k+1}\|})/\log(\frac{\|x_{k+2}-x_{k+1}\|}{\|x_{k+1}-x_{k+2}\|})$. The sequence $\{x_k\}$ is quadratic convergent if $\lim_{k\to\infty} q_k$=2 [12]. The curves of $\{q_k\}$ is shown in Fig 3.

Fig. 3 shows that q_k converges to 2, accordingly, the dotted curves in Fig 2 are quadratic convergent.

VI. Conclusion

The convergence analysis of Newton-Raphson method in feasible power-flow for DC network is presented in this letter. Firstly, the solvability of the power-flow equation is analyzed and the inherent property of the solution is obtained. Then, the monotonicity of the NR method is analyzed, consequently, a concise and less conservative convergence condition is obtained, which can provide an easier way to set the initial value to guarantee the convergence of the Newton-Raphson method.

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